Some games and their topological consequences III

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ICMC-USP (Partially supported by FAPESP)

Baire spaces

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Definition

A topological space is **Baire** if for every family $\langle A_n : n \in \omega \rangle$ of open dense subsets, $\bigcap_{n\in\omega} A_n$ is dense.

A classical game

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- ALICE plays A_0 a non-empty open set;
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- an so on, for every $n \in \omega$.

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- an so on, for every $n \in \omega$.

At the end, BOB is declared the winner if $\bigcap_{n\in\omega} B_n \neq \emptyset$ and ALICE is the winner otherwise.

A classical relation

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Theorem (Oxtoby)
X is a Baire space if and only if ALICE *does not have a winning strategy for the Banach-Mazur game.*

Warming up

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- Alice plays $V \cap A_0$;
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$\bigcup_{B \in \tau_{\subset A}} \sigma(\langle B \rangle)$ *is open dense in* V*.*

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Note that the above lemma just tells us that $\bigcup_{A \in S_1} A$ is open dense in V.

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And basically with the same proof, $D_n = \bigcup_{A \in S_n} A$ is open dense in V for every n.

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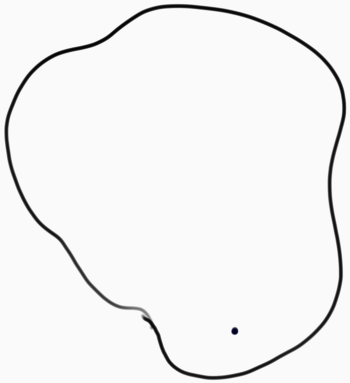
Now BOB just has to follow this x. At the inning n, BOB just picks a open set that has x in its interior. Since x is in the intersection, the answer from ALICE will also contain x. ?

We may have a problem here.

What can happen?



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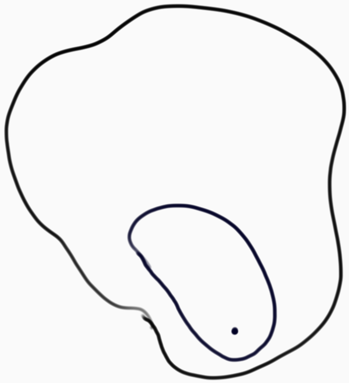

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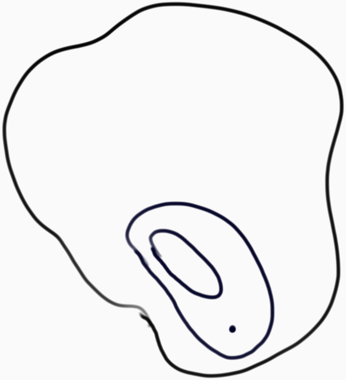

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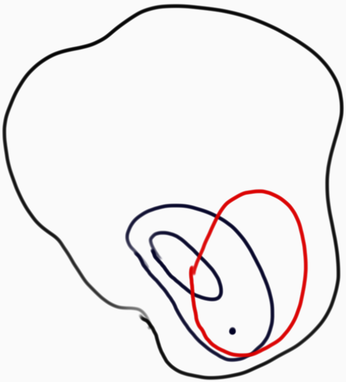

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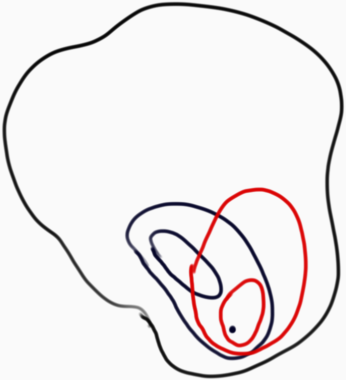

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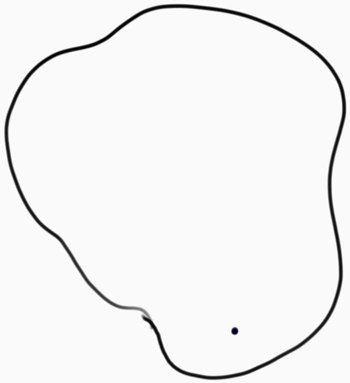

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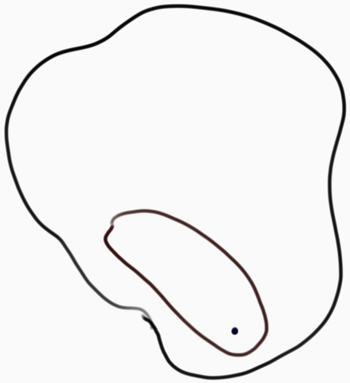

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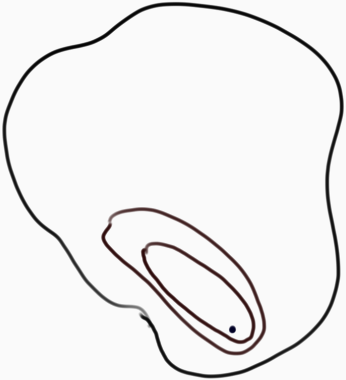

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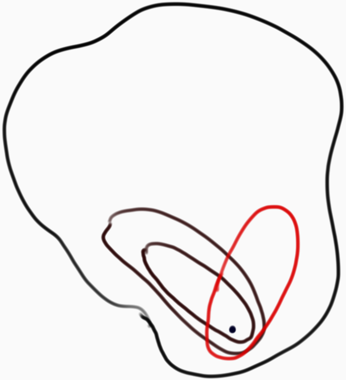

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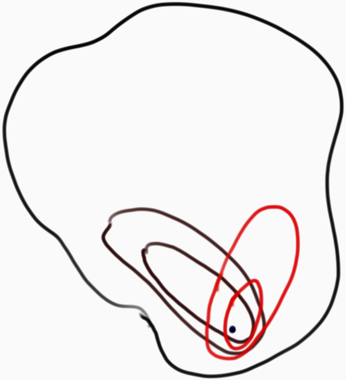

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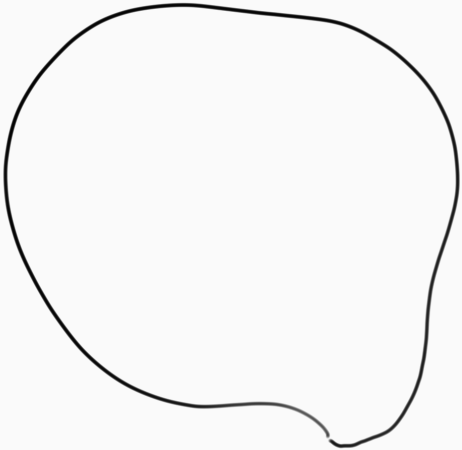
Instead of just looking for the possible answers, we look for maximal antichains (and one being a refinement of the previous one).

It is better if we draw a picture



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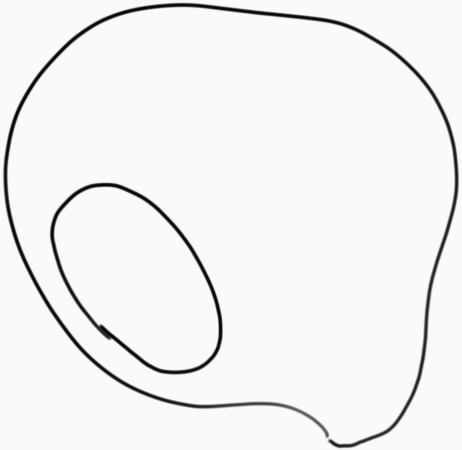

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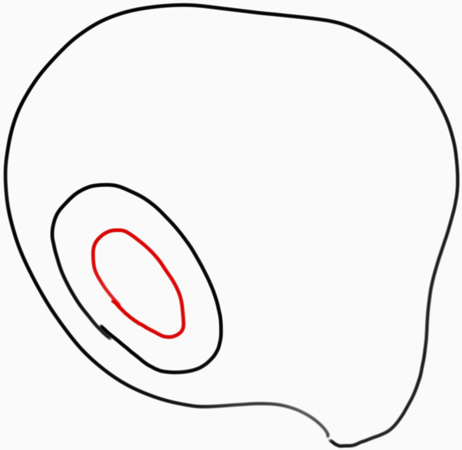

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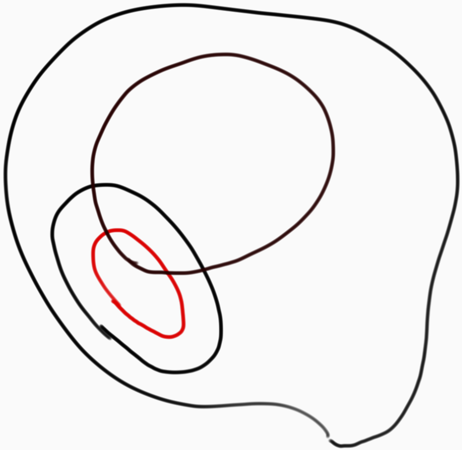

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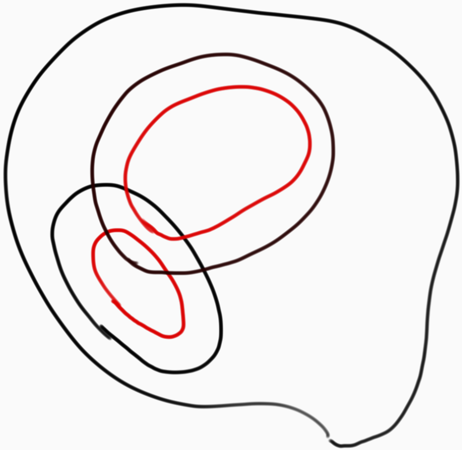

Products

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Theorem

There are Baire spaces X and Y such that $X \times Y$ is not Baire.

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Theorem

If Bob *has a winning strategy for the Banach-Mazur game, then the space is productively Baire.*

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First note that we can suppose that ea always plays basic open sets.

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We go back to $X \times Y$ and let $V_1 \times W_1 = ea(\langle B_0 \times U_0 \rangle)$.

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The point is, $\bigcap_{n\in\omega} B_n$ is non-empty. $\bigcap_{n\in\omega} B_n \times U_n = \emptyset$.

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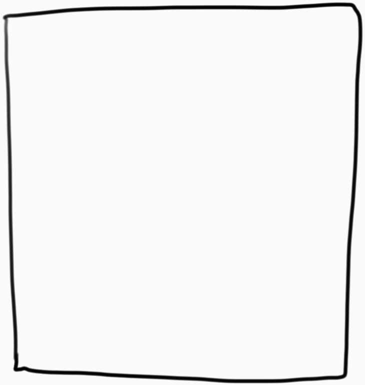
Start over.

The point is, $\bigcap_{n\in\omega} B_n$ is non-empty. $\bigcap_{n\in\omega} B_n \times U_n = \emptyset$. So $\bigcap_{n\in\omega} W_n = \emptyset$.

It is not so confusing



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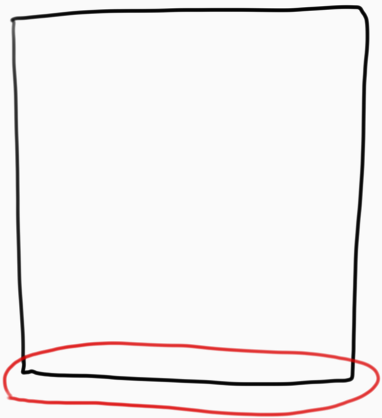

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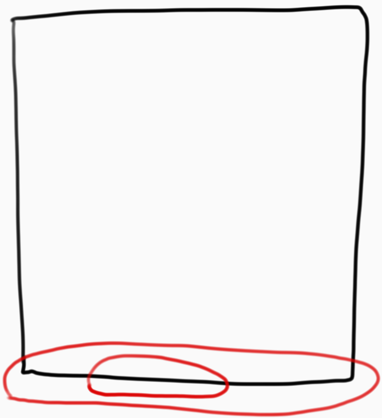

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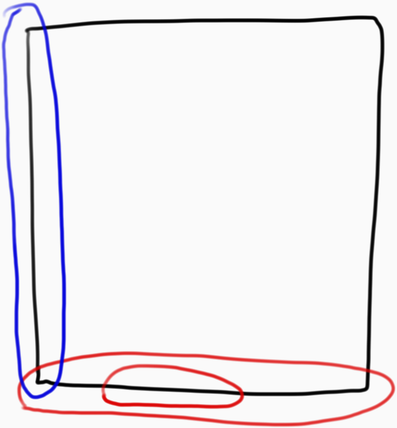

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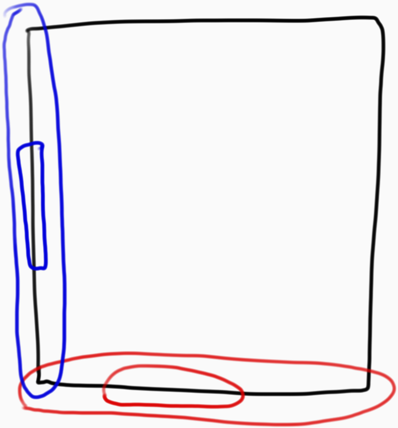

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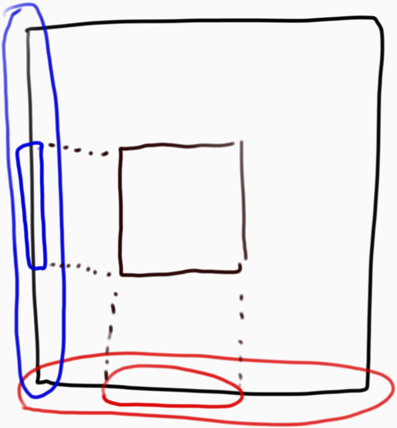

Cantor

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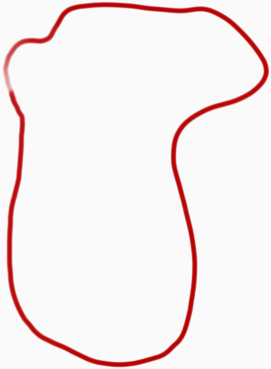
Proposition

Let $X \subset \mathbb{R}$. If $\textsc{Bob}$ *has a winning strategy for the Banach-Mazur game over X, then X has a Cantor subspace.*

Parallel realities



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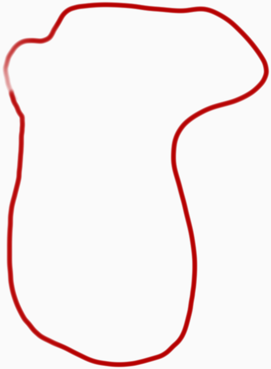

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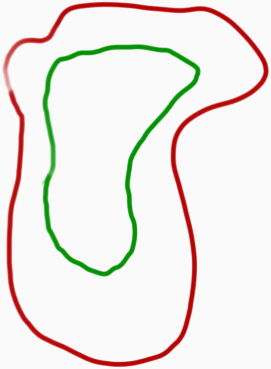

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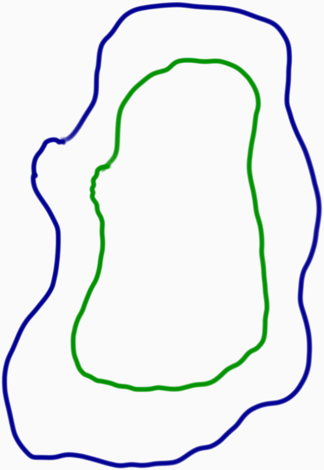

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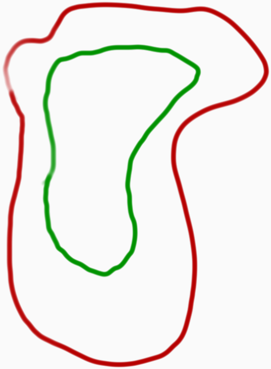

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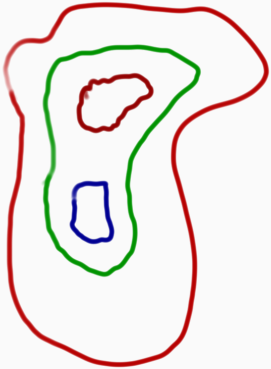

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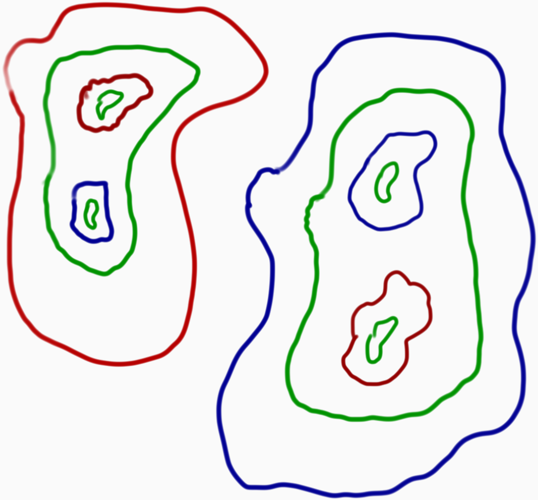

Bernstein

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Definition

We say that $X \subset \mathbb{R}$ is a **Bernstein set** if it is uncountable and, for every uncountable closed set $F \subset X$, $F \cap X$ e $F \cap (\mathbb{R} \setminus X)$ are both non-empty.

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Corollary

If X is a Bernstein set, then Bob *has no winning strategy.*

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- Bob plays $B_1^1, B_1^2 \subset A_1^1$ and $B_1^3, B_1^4 \subset A_1^2$ non-empty open sets;

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Bob is declared the winner if $\bigcap_{n \in \omega} B_n \neq \emptyset$ and Alice is the winner otherwise.

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Are these games different?

Bernstein again

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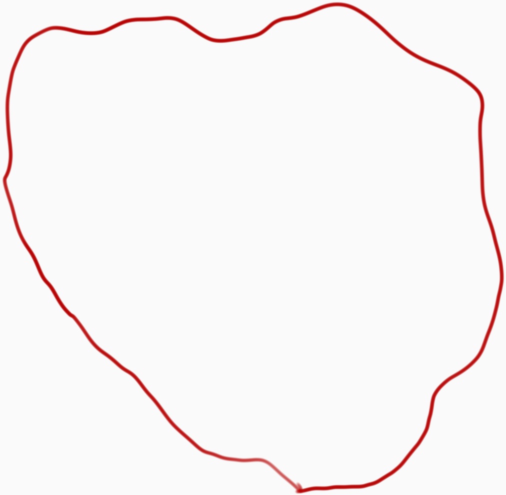
Proposition

If X is a Bernstein set, then BOB *has a winning strategy for this new game.*

Think that you are playing over $\mathbb{R}$



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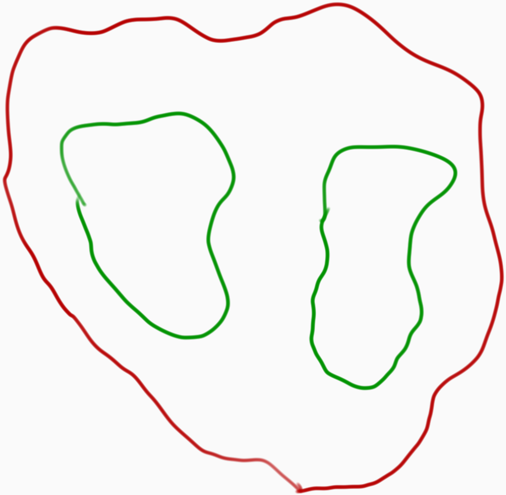

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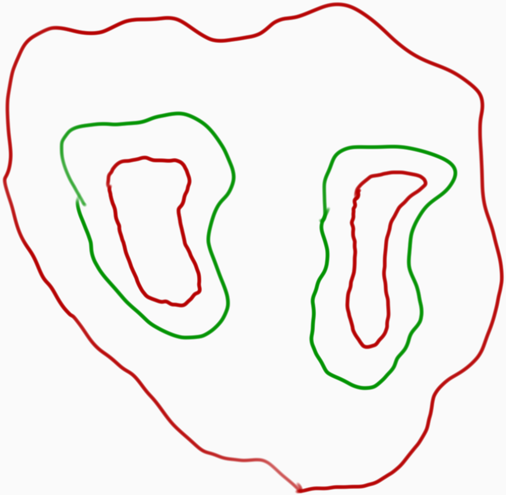

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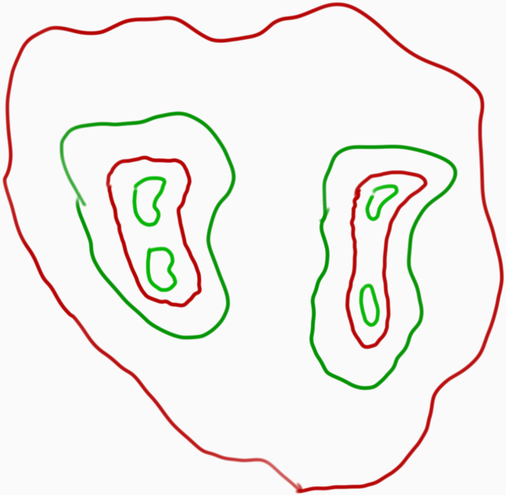

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Multiboard game

Let BM^2 be the 2-boards game version of Banach-Mazur. There are two boards of the game, Alice starts playing on all the boards. Then Bob answers playing in all the boards (following the rules on each board). Then Alice again and so on.

We say that Bob wins the game if he wins on all boards.

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Really multiboard game

Let BM^κ be the κ-boards game version of Banach-Mazur. There are κ boards of the game, Alice starts playing on all the boards. Then Bob answers playing in all the boards (following the rules on each board). Then Alice again and so on.

We say that Bob wins the game if he wins on all boards. Alice is the winner otherwise (i.e. Alice wins at some board).

Seeing the games

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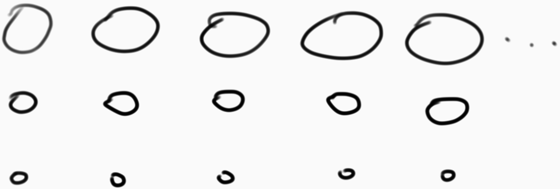

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Bibliography

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